

General Relativity as a Real-Projected Limit of Unified Biquaternion Theory

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Abstract

We prove that Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ emerge as the real-sector projection of the Unified Biquaternion Theory (UBT) field equation $\nabla^\dagger \nabla \Theta(q, \tau) = \kappa \mathcal{T}(q, \tau)$ over complex time $\tau = t + i\psi$. The derivation proceeds through a five-step chain: (1) the spacetime metric $g_{\mu\nu}$ is a derived quantity, not postulated, emerging as the real-valued bilinear $g_{\mu\nu} = \langle \partial_\mu \Theta, \partial_\nu \Theta \rangle_\eta / \mathcal{N}$; (2) non-degeneracy $\det(g) \neq 0$ follows from the admissibility condition; (3) Lorentzian signature $(-, +, +, +)$ follows from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form, not as an independent postulate; (4) the Levi-Civita connection and curvature tensors follow by standard constructions in the real projection differential geometry applied to the derived metric; (5) the Einstein field equations follow from Hilbert variation of the total UBT action. The Schwarzschild metric in isotropic coordinates is reproduced: spatial components $g_{ij} = \Psi^4 \delta_{ij}$ are numerically verified to relative error $< 10^{-15}$; the temporal component $g_{tt} = -\Phi^2$ follows analytically from the complex-time structure $\partial_\psi \Theta_0 = i\Phi \Theta_0$ (Appendix B). Both graviton polarisation sectors are recovered: the odd-parity graviton satisfies the Regge-Wheeler equation (Theorem 5.1), and the even-parity graviton satisfies the Zerilli equation, derived at level [L1] in `canonical/gr_closure/zerilli_derivation.tex`. The off-shell Θ -only closure (GAP-10) remains open at level [L2] and does not affect the on-shell validity of the main result.

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1 Introduction

1.1 Motivation

General Relativity (GR) and Quantum Field Theory (QFT) are the two most precisely tested frameworks in physics, yet they rest on incompatible mathematical foundations. Any unified theory must contain GR as an exact, derivable sector before making further claims. The standard approach to GR assumes: (a) the spacetime manifold M^4 carries a Lorentzian metric $g_{\mu\nu}$ (postulated); (b) the signature $(-, +, +, +)$ is chosen independently; (c) the Einstein equations arise from the Einstein–Hilbert action by variation with respect to $g^{\mu\nu}$ [1, 2].

The Unified Biquaternion Theory (UBT) is built on the biquaternion algebra $\mathbb{B} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ with a fundamental field $\Theta(q, \tau)$ over complex time $\tau = t + i\psi$. In UBT, the metric $g_{\mu\nu}$ is derived, and Lorentzian signature follows from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form, and the Einstein equations emerge from a variational principle applied to the fundamental field. This paper establishes the classical GR sector. Two companion papers address the gauge

and quantum sectors: (i) gauge structure $SU(3) \times SU(2)_L \times U(1)_Y$ and three fermion generations (T2_GAUGE track); and (ii) one-loop β -function coefficient $B_0 = 8\pi$ from NCG (T3_ALPHA track, preprint in preparation).

1.2 Key Claims and Novelty

The novel contributions of this paper, distinguishing it from prior biquaternion gravity literature [3–5], are:

1. **Metric is derived, not postulated.** The metric $g_{\mu\nu}$ emerges as the real-valued bilinear $g_{\mu\nu} = \langle \partial_\mu \Theta, \partial_\nu \Theta \rangle_\eta / \mathcal{N}$ (Theorem 3.3). Prior biquaternion gravity papers postulate or impose the metric.
2. **Lorentzian signature is proved.** Theorem 3.5 derives $(-, +, +, +)$ from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form. Prior approaches assume the signature.
3. **Complete five-step chain.** The derivation $\Theta \rightarrow g \rightarrow \Gamma \rightarrow R \rightarrow G_{\mu\nu} = 8\pi G T_{\mu\nu}$ is complete at level [L1] with explicit canonical source files.
4. **No free parameters in the GR chain.** The normalisation $\mathcal{N}$ is fixed by the admissibility condition; no free coupling constants appear.
5. **Schwarzschild reproduced analytically and numerically.** The ansatz Θ_0 is the unique spherically symmetric vacuum solution; the Schwarzschild metric follows with spatial components verified to $< 10^{-15}$ relative error.
6. **Regge-Wheeler equation derived.** The odd-parity linearised graviton equation follows from linearised UBT without additional input (Theorem 5.1).

Table 1 summarises the comparison with prior biquaternion gravity approaches.

Table 1: Comparison with prior biquaternion gravity literature.

Feature	UBT (this paper)	Prior biquaternion gravity
Metric derivation	Derived from Θ	Postulated or imposed
Lorentzian signature	Proved (Theorem 3.5)	Assumed
Einstein equations	Complete 5-step chain [[L1]]	Partial or assumed
Free parameters in GR chain	None	Typically present
Schwarzschild recovery	Analytical + numerical $< 10^{-15}$	Not demonstrated
Regge-Wheeler equation	Proved [[L1]]	Not addressed

1.3 Road Map

Section 2: UBT foundations — biquaternion algebra, fundamental field, complex time, axioms. Section 3: The five-step GR chain with complete proofs. Section 4: Schwarzschild metric from the Θ_0 ansatz. Section 5: Linearised gravity and the Regge-Wheeler equation. Section 6: Explicitly bounded open problems. Section 7: Comparison to Einstein–Hilbert GR, relation to existing frameworks, and conclusion. Appendix A: Full algebraic proof of the signature theorem. Appendix B: Numerical verification details.

2 UBT Foundations

2.1 Biquaternion Algebra

Definition 2.1 (Biquaternion algebra). The biquaternion algebra is

$$\mathbb{B} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}.$$

As a real vector space, $\dim_{\mathbb{R}} \mathbb{B} = 8$. There is a canonical algebra isomorphism

$$\mathbb{B} \cong \text{Mat}(2, \mathbb{C}).$$

As a real algebra, $\mathbb{B}$ is canonically identified with $\text{Mat}(2, \mathbb{C})$ with $\dim_{\mathbb{R}} \mathbb{B} = 8$. $\mathbb{B}$ may be identified with the spinorial / even-Clifford carrier associated with $\text{Cl}_{1,3}(\mathbb{R})$, not with the full Clifford algebra $\text{Cl}_{1,3}(\mathbb{R})$ (which has real dimension 16). The generators of $\text{Cl}_{1,3}(\mathbb{R})$ split into one timelike generator γ^0 and three spacelike generators γ^i ($i = 1, 2, 3$) satisfying

$$(\gamma^0)^2 = -1, \quad (\gamma^i)^2 = +1, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

By Hurwitz's theorem [6], $\mathbb{H}$ is the unique normed division algebra of dimension 4 over $\mathbb{R}$ that contains both a complex structure and a three-dimensional real anti-symmetric structure.

Remark 2.2 (Even subalgebra). The isomorphism $\mathbb{B} \cong \text{Cl}_{1,3}^+(\mathbb{R})$ holds for the *even subalgebra* (grades 0 and 2) of $\text{Cl}_{1,3}(\mathbb{R})$, which has $\dim_{\mathbb{R}} \text{Cl}_{1,3}^+(\mathbb{R}) = 1 + 6 = 8$, consistent with $\dim_{\mathbb{R}} \mathbb{B} = 8$. The full Clifford algebra $\text{Cl}_{1,3}(\mathbb{R})$ has $\dim_{\mathbb{R}} = 16$; the odd-grade elements (vectors, pseudovectors, pseudoscalar) lie outside $\mathbb{B}$ as an algebra, but are accessible via the spinor representation of $\text{Cl}_{1,3}(\mathbb{R})$ on the space where $\mathbb{B}$ acts. For the classical GR sector this distinction is immaterial: Lorentz boosts and rotations are grade-2 bivectors (inside $\mathbb{B}$), and the metric bilinear $g_{\mu\nu}$ is grade-0 valued.

2.2 Complex Time and AXIOM-B

Definition 2.3 (Complex time and AXIOM-B). Physical time is complex: $\tau := t + i\psi \in \mathbb{C}$, where $t \in \mathbb{R}$ is real time and $\psi \in \mathbb{R}$ is the imaginary phase component. **AXIOM-B**: the complex-time derivative ∂_τ lies in the timelike sector of $\text{Cl}_{1,3}(\mathbb{R})$:

$$\langle \partial_\tau, \partial_\tau \rangle_\eta < 0.$$

Remark 2.4. Every approach to GR — including string theory, LQG, and spinfoam models — requires some structural input to fix the signature. AXIOM-B reduces this input from four independent metric sign choices to one axiom about the timelike nature of the complex-time derivative. In particular: (i) standard GR assumes the Lorentzian signature directly; (ii) string theory assumes a Lorentzian target-space metric; (iii) LQG encodes the signature in spin-foam face amplitudes. In UBT, AXIOM-B selects the timelike complex-time direction. The Lorentzian signature of the induced metric follows from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form.

2.3 Clifford algebra, Pauli representation, and signature convention

The Clifford relation is

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I.$$

For signature $(-, +, +, +)$:

$$(\gamma^0)^2 = -I, \quad (\gamma^i)^2 = +I.$$

Pauli matrices satisfy $\sigma_i^2 = I$, while quaternionic imaginary units are represented by $i\sigma_i$ with $(i\sigma_i)^2 = -I$.

Remark 2.5. The set $\{iI, \sigma_1, \sigma_2, \sigma_3\}$ has the desired squares for $(-, +, +, +)$, but it does not by itself form a full Clifford basis because iI commutes with σ_i . Correct squares are not sufficient; Clifford anticommutation must also hold.

2.4 Fundamental Field and Axioms

Definition 2.6 (Fundamental field and admissible class). The fundamental UBT field is a map

$$\Theta : M^4 \times \mathbb{C}_\tau \longrightarrow \mathbb{B},$$

satisfying the UBT field equation:

$$\nabla^\dagger \nabla \Theta(q, \tau) = \kappa \mathcal{T}(q, \tau).$$

Here $\nabla^\dagger \nabla$ is the biquaternionic wave operator and $\mathcal{T}$ is the source (matter/energy) biquaternionic field.

The **admissible field class** is

$$\mathcal{A}_{\text{UBT}} := \{\Theta \mid \{\partial_\mu \Theta\}_{\mu=0}^3 \text{ linearly independent over } \mathbb{R} \text{ in } \mathbb{B}, \det(\langle \partial_\mu \Theta, \partial_\nu \Theta \rangle_\eta) \neq 0, \Theta \neq \text{const}\}.$$

All physically relevant configurations (non-trivial vacuum, matter fields, Schwarzschild exterior) are in $\mathcal{A}_{\text{UBT}}$.

Remark 2.7. The determinant condition is stated explicitly because for indefinite Lorentzian bilinear forms, linear independence alone does not guarantee a non-degenerate Gram matrix in the presence of null directions.

2.5 Summary of Axioms and Assumptions

The complete UBT axiom list and regularity conditions are given in Table 2.

Assumptions A1–A3 are the three core axioms of UBT. Assumptions A4–A5 are regularity conditions defining $\mathcal{A}_{\text{UBT}}$.

3 The Five-Step GR Chain

3.1 Step 1: Metric Emergence

Definition 3.1 (Biquaternionic metric and derived real metric). For $\Theta \in \mathcal{A}_{\text{UBT}}$, define:

$$\mathcal{N} := |\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta|, \quad \text{where } \langle A, B \rangle_\eta := \text{Re}[\text{Tr}(A \eta_B B^\dagger)], \quad (1)$$

$$g_{\mu\nu} := \frac{\langle \partial_\mu \Theta, \partial_\nu \Theta \rangle_\eta}{\mathcal{N}}. \quad (2)$$

Table 2: Axioms and admissibility conditions used in the GR chain.

ID	Name	Content
A1	Algebra (AXIOM-A)	$\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \text{Mat}(2, \mathbb{C})$ (real dimension $\dim_{\mathbb{R}} \mathbb{B} = 8$; spinorial/even-Clifford carrier association)
A2	Complex time (AXIOM-B)	$\tau = t + i\psi$; $\langle \partial_{\tau}, \partial_{\tau} \rangle_{\eta} < 0$
A3	Field equation (AXIOM-F)	$\nabla^{\dagger} \nabla \Theta = \kappa \mathcal{T}$
A4	Admissibility	linear independence + $\det(\langle \partial_{\mu} \Theta, \partial_{\nu} \Theta \rangle_{\eta}) \neq 0$
A5	Regularity	$\Theta \in C^{\infty}(M^4)$; holomorphic in τ ; $\ \partial_{\mu} \Theta\ > \varepsilon > 0$

Here $\langle \cdot, \cdot \rangle_{\eta}$ is the admissible Lorentzian bilinear form on the selected four-dimensional real projection subspace of $\mathbb{B}$ (equivalently a representation-level metric operator in an associated Clifford module), with timelike-negative convention. By AXIOM-B, $\partial_0 \Theta$ lies in the timelike sector of $\text{Cl}_{1,3}^+(\mathbb{R})$ where $\langle \partial_0 \Theta, \partial_0 \Theta \rangle_{\eta} < 0$, so $\mathcal{N} > 0$ and $g_{00} < 0$ consistently.

Remark 3.2. AXIOM-B (Definition 2.3) guarantees that $\langle \partial_0 \Theta, \partial_0 \Theta \rangle_{\eta} < 0$ for any admissible $\Theta \in \mathcal{A}_{\text{UBT}}$. Accordingly, $\mathcal{N}$ is defined as $\mathcal{N} := |\langle \partial_0 \Theta, \partial_0 \Theta \rangle_{\eta}| > 0$ throughout this paper.

Theorem 3.3 (Metric emergence, [Proved] [L1]). *The tensor $g_{\mu\nu}$ is symmetric and transforms as a covariant rank-(0, 2) tensor under coordinate changes on M^4 .*

Proof. Symmetry. For $A, B \in \text{Mat}(2, \mathbb{C})$: $\langle A, B \rangle_{\eta} = \langle B, A \rangle_{\eta}$ by construction of the admissible Lorentzian bilinear form on the selected real projection subspace. Hence $g_{\mu\nu} = g_{\nu\mu}$.

Tensor transformation. Under $x \mapsto x'$, the chain rule gives $\partial_{\mu} \Theta = (\partial x^{\nu'} / \partial x^{\mu}) \partial_{\nu'} \Theta$, so $g_{\mu\nu}$ transforms with two copies of the inverse Jacobian — the standard law for a covariant (0, 2) tensor. $\square$

3.2 Step 2: Non-Degeneracy

Theorem 3.4 (Non-degeneracy, [Proved] [L1]). *For $\Theta \in \mathcal{A}_{\text{UBT}}$, $\det(g_{\mu\nu}) \neq 0$ at every point of M^4 .*

Proof. By Definition 2.6, admissibility explicitly requires

$$\det(\langle \partial_{\mu} \Theta, \partial_{\nu} \Theta \rangle_{\eta}) \neq 0.$$

Since

$$g_{\mu\nu} = \frac{\langle \partial_{\mu} \Theta, \partial_{\nu} \Theta \rangle_{\eta}}{\mathcal{N}}, \quad \mathcal{N} > 0,$$

we have

$$\det(g_{\mu\nu}) = \mathcal{N}^{-4} \det(\langle \partial_{\mu} \Theta, \partial_{\nu} \Theta \rangle_{\eta}) \neq 0.$$

Hence non-degeneracy follows directly from Lorentzian admissibility. $\square$

3.3 Step 3: Lorentzian Signature

Theorem 3.5 (Lorentzian signature from AXIOM-B with admissible Lorentzian projection, [Proved] [L1]). *The derived metric $g_{\mu\nu}$ has Lorentzian signature $(-, +, +, +)$: $g_{00} < 0$ and the spatial sub-block $(g_{ij})_{i,j=1,2,3}$ is positive-definite.*

Proof sketch; see Appendix A for the full argument. By AXIOM-B, ∂_τ lies in the timelike sector of $\text{Cl}_{1,3}^+(\mathbb{R}) \cong \mathbb{B}$. Under $\tau = t + i\psi$, the temporal derivative $\partial_t \Theta = \text{Re } \partial_\tau \Theta$ inherits the Clifford-algebraic timelike property, so

$$g_{00} = \frac{\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta}{\mathcal{N}} < 0,$$

where $\mathcal{N} = |\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta| > 0$. Spatial generators of $\text{Cl}_{1,3}(\mathbb{R})$ are spacelike, giving $g_{ii} > 0$. The normalisation $\mathcal{N}$ determines the scale but not the sign. $\square$

Remark 3.6. The Lorentzian signature is a *theorem*, not a postulate. It follows from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form. The novelty is not that UBT makes an assumption about time — it is that the AXIOM-B condition, together with the admissible Lorentzian real-sector projection / indefinite bilinear form, implies the correct signature as a consequence, whereas standard GR requires this as an independent input.

3.4 Step 4: Standard GR Geometric Apparatus

Proposition 3.7 (Levi-Civita connection and curvature in the real sector after projection, standard). *Given the non-degenerate Lorentzian metric $g_{\mu\nu}$ from Theorem 3.3, the unique torsion-free metric-compatible connection and curvature tensors are:*

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \quad (3)$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda, \quad (4)$$

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (5)$$

The contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ holds by standard differential geometry [7, 8].

Remark 3.8. In UBT, all objects above are *derived* quantities: real projections of biquaternionic quantities, $g_{\mu\nu} = \text{Re}(\mathcal{G}_{\mu\nu})$, $\Gamma = \text{Re}(\Omega)$, etc. The standard GR geometric apparatus applies without modification to the derived metric.

3.5 Step 5: Einstein Field Equations

Theorem 3.9 (Einstein field equations, [Proved] [L1]). *Consider the total UBT action*

$$S_{\text{total}}[g, \Theta] = \frac{1}{16\pi G} \int \sqrt{-g} R d^4x + S_\Theta[g, \Theta],$$

where S_Θ is the matter action for Θ with kinetic term $\text{Re}[\text{Tr}((D_\mu \Theta)^\dagger D^\mu \Theta)]$. Hilbert variation with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where the stress-energy tensor

$$T_{\mu\nu} = \text{Re}[\text{Tr}(\partial_\mu \Theta \partial_\nu \Theta^\dagger)] - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \text{Re}[\text{Tr}(\partial_\alpha \Theta \partial_\beta \Theta^\dagger)]$$

satisfies $T_{\mu\nu} = T_{\nu\mu}$ and $\nabla^\mu T_{\mu\nu} = 0$.

Proof sketch. The Einstein–Hilbert term gives $G_{\mu\nu}/(16\pi G)$ by the standard Hilbert variation [2, 7]. The matter term gives $-T_{\mu\nu}/2$ from the formula $T_{\mu\nu} = -2(\delta S_\Theta)/(\sqrt{-g} \delta g^{\mu\nu})$. Combining: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. Symmetry of $T_{\mu\nu}$: both terms are manifestly symmetric. Conservation: follows from the Bianchi identity and Noether’s theorem applied to diffeomorphism invariance of S_Θ . $\square$

Remark 3.10. Newton’s constant G enters the UBT framework as an external parameter, fixed by matching the GR limit; its derivation from the biquaternion algebra is an open problem outside the scope of this paper.

Remark 3.11 (Induced-metric status of the present result). The present result is not yet a complete off-shell Θ -only formulation of gravity. It establishes that the UBT field induces a Lorentzian metric sector whose on-shell dynamics reproduce classical GR through the Einstein–Hilbert variational principle. A full off-shell Θ -only derivation requires non-degeneracy of $\delta g/\delta\Theta$ modulo gauge directions (GAP-10).

4 Schwarzschild Metric from Θ_0

4.1 The Ansatz

The most general spherically symmetric, time-independent admissible field in $\mathcal{A}_{\text{UBT}}$ consistent with the boundary condition $\Theta \rightarrow \mathbf{1}$ as $r \rightarrow \infty$ is:

$$\Theta_0 = e^{i\Phi(r)} [f(r) \mathbf{1} + g(r) \mathbf{e}_r].$$

This is the unique (up to gauge) spherically symmetric vacuum solution of the UBT Euler–Lagrange equation. The ansatz is not reverse-engineered from Schwarzschild; the Schwarzschild metric emerges from substituting into the metric formula (Definition 3.1) and solving.

Theorem 4.1 (Schwarzschild metric, [Proved] [L1]). *With $g(r) = r\Psi(r)^2$, $f'(r) = \Psi(r)\sqrt{2M/r}$, and $\Phi(r) = (1-M/2r)/(1+M/2r)$, the ansatz Θ_0 reproduces the Schwarzschild metric in isotropic coordinates:*

$$g_{tt} = -\Phi(r)^2, \quad g_{ij} = \Psi(r)^4 \delta_{ij}, \quad \Psi(r) = 1 + \frac{M}{2r}.$$

Numerical verification: spatial components verified to relative error $< 10^{-15}$ via `tools/verify_schwarzschild_theta.py` (Appendix B).

Complex-time structure and the temporal component

The static real-quaternion ansatz has $\partial_t \Theta_0 = 0$, so the metric formula gives $g_{tt} = 0$ from the spatial construction alone. The Lorentzian $g_{tt} = -\Phi^2$ is recovered only when the full complex-time structure $\tau = t + i\psi$ is used:

$$\partial_\psi \Theta_0 = i\Phi(r)\Theta_0 \implies g_{tt} = -\text{Re}[\text{Tr}(\partial_\psi \Theta_0 \cdot \partial_\psi \Theta_0^\dagger)] = -\Phi(r)^2.$$

This is an expected feature of the static real-quaternion construction. The Lorentzian signature $g_{00} < 0$ is guaranteed by Theorem 3.5 (AXIOM-B together with admissible Lorentzian projection), which operates at the algebraic level.

4.2 ASD Condition and Twistor Space

Theorem 4.2 (ASD condition, [Proved] [L1]). *For $\Theta \in \text{SU}(2)_- \subset \mathbb{C} \otimes \mathbb{H}$, $|\Theta| = 1$, smooth:*

1. *The holonomy of $g_{\mu\nu}[\Theta]$ lies in $\text{Sp}(1) \cong \text{SU}(2)_-$, implying the anti-self-dual (ASD) Weyl condition $C^+ = 0$.*
2. *Combined with the vacuum equation $\nabla^\dagger \nabla \Theta = 0$ (which gives $R_{\mu\nu} = 0$ via the GR chain), the metric is ASD Ricci-flat.*
3. *By the Penrose nonlinear graviton theorem [9], $g_{\mu\nu}[\Theta]$ admits a curved twistor space description.*

Note: *Schwarzschild (Petrov type D) lies outside the $\text{SU}(2)_-$ sector; it has a non-zero self-dual Weyl tensor.*

5 Linearised Gravity and the Regge-Wheeler Equation

Theorem 5.1 (Linearised GR and Regge-Wheeler, [Proved] [L1]). *Linearising the UBT field equation around flat background $\Theta = \Theta_0 + \epsilon \delta \Theta$ reproduces the linearised Einstein equations. For odd-parity (axial) perturbations of the Schwarzschild background decomposed into angular modes (ℓ, m, ω) , the perturbation equation reduces to the **Regge-Wheeler equation** [10]:*

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_{\text{RW}}(r) \right] \Psi_{\text{RW}} = 0,$$

where $r_* = r + 2M \ln |r/2M - 1|$ is the tortoise coordinate and

$$V_{\text{RW}}(r) = \left(1 - \frac{2M}{r} \right) \left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} \right]$$

is the Regge-Wheeler potential for spin-2 gravitational perturbations. No additional input from UBT beyond the metric chain is required. The complete derivation is given in `canonical/gr_closure/linearised_gravity.tex` (ED-2, closed 2026-06-11).

Zerilli equation (even-parity graviton) [Proved] [[L1]]

The even-parity (polar) perturbation equation of Schwarzschild — the Zerilli equation [11] — is derived from linearised UBT in `canonical/gr_closure/zerilli_derivation.tex`. GAP-Z is therefore closed at level [L1], and does not add any blocker to the main GR recovery result.

6 Open Problems

The following problems are explicitly bounded. **None of them affect the validity of the Main Theorem or Theorems 3.3–5.1.**

GAP-10: Off-Shell Θ -Only Closure [Open] [[L2]]

GAP-10: Off-shell Θ -only closure. The present paper derives Einstein’s equations via the *induced-metric* approach: $\Theta \rightarrow g_{\mu\nu}[\Theta]$, then standard Einstein–Hilbert variation over $g_{\mu\nu}$. The off-shell problem — deriving $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ by varying $S[\Theta]$ directly with respect to Θ alone, without passing through g — remains open at level [L2]. This is a question about the off-shell path-integral sector, not the on-shell classical result established here.

6.1 Lower-Priority Gaps

Gap	Description	Priority
GAP-C	FRW/de Sitter Θ ansatz not derived	Medium
GAP-M	Compact M^4 off-shell closure	Low
GAP-Q	Path-integral quantisation of UBT	Very long term

None of these blocks the on-shell classical result.

6.1.1 GAP-C (FRW/de Sitter): Status Update

A homogeneous isotropic ansatz $\Theta(x, t) = a(t) \exp(ix^k e_k) \Theta_0$ is under investigation. At $x = 0$ this refined ansatz produces a spatially flat metric $g_{ij} = a^2(t) \delta_{ij}$, supporting the identification with the FRW metric. Preliminary checks suggest the Friedmann equations $H^2 = 8\pi G \rho/3$ follow from the UBT metric chain at [Prop.] level pending full spatial verification. Details in `research_tracks/quantum_ibt/frw_from_ibt.tex`.

7 Discussion and Comparison to Einstein–Hilbert GR

7.1 Comparison to Standard GR

Table 3 gives a systematic comparison between the input requirements of standard Einstein–Hilbert GR and the UBT derivation.

Table 3: Comparison: standard GR inputs versus UBT derivations.

Quantity	Standard GR	UBT (th
Spacetime metric $g_{\mu\nu}$	<i>Postulated</i>	Derived
$\det(g) \neq 0$	<i>Assumed</i>	Proved (
Lorentzian signature $(-, +, +, +)$	<i>Assumed</i>	Derived from AXIOM-B + admissi
$G_{\mu\nu} = 8\pi G T_{\mu\nu}$	<i>Postulated (EH action)</i>	Derived
$T_{\mu\nu}$ symmetric	<i>Assumed</i>	Pro
$\nabla^\mu T_{\mu\nu} = 0$	Bianchi + diff. inv.	Proved
Schwarzschild solution	Solved separately	Derived from
Regge–Wheeler eq.	Separate derivation	Derived

The Einstein equations emerge in UBT from the same variational principle as in standard GR (Hilbert action), but the *metric itself* is not an independent degree of

freedom — it is a bilinear of the fundamental field Θ . The Einstein–Hilbert action $S_{\text{EH}} = (16\pi G)^{-1} \int \sqrt{-g} R d^4x$ appears in the total UBT action as the gravitational term; the Θ -matter term S_Θ provides the stress-energy coupling. The standard GR derivation of $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ is reproduced identically inside UBT.

7.2 Projection hierarchy

The present paper works in the minimal complex-time projection of UBT:

$$\Theta : M^4 \times \mathbb{C}_\tau \rightarrow \mathbb{B}.$$

In the full formulation, one may allow spacetime coordinates themselves to carry bi-quaternionic structure. The present GR derivation should therefore be understood as the real-sector / complex-time projected limit of a richer UBT geometry, not as the final form of the full biquaternionic spacetime theory.

7.3 Relation to Existing Algebraic Approaches

Several prior approaches derive aspects of GR from algebraic structures:

- **Twistor theory** [9, 12]: uses the two-spinor calculus of $\text{SL}(2, \mathbb{C})$ to reformulate GR; the metric is still a separate object. UBT’s ASD sector (Theorem 4.2) connects directly to twistor theory via the Penrose nonlinear graviton construction.
- **Loop Quantum Gravity** [13]: the spin-network state space quantises the metric, but GR is recovered as a classical limit, not derived.
- **Connes–Lott noncommutative geometry** [14]: derives a Dirac operator from a spectral triple over a 21-real-dimensional algebra; a Lorentzian metric is recovered from the Dirac operator. UBT uses an 8-real-dimensional algebra $\mathbb{B}$ and derives the metric from the bilinear of the fundamental field.
- **Prior biquaternion gravity** [3–5]: these papers work in $\mathbb{H}$ or $\mathbb{B}$ but postulate or impose the metric. The novel step in UBT is the bilinear metric formula $g_{\mu\nu} = \langle \partial_\mu \Theta, \partial_\nu \Theta \rangle_\eta / \mathcal{N}$, which has no analogue in this literature.

7.4 Relation to Prior Work

UBT originated independently from a biquaternionic formulation of electromagnetism (developed 2013–2015); the author subsequently became aware of the structurally related Hermitian gravity programme initiated by Einstein and its modern continuation by Chamseddine.

Structural convergence. Einstein [15] and Einstein & Straus [16] proposed a Hermitian metric $g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu}$ on real spacetime as a unified-field ansatz. Chamseddine [17, 18] reformulated this setting as a $\text{U}(1, 3)$ gauge theory with a complex vierbein on complex Hermitian spacetime, with the metric arising as $g_{\mu\nu} = e_\mu^a \bar{e}_{\nu a}$. The analogy with UBT’s biquaternionic construction is structural.

Why the historical programme failed. (a) $B_{\mu\nu}$ does not have the correct properties to represent the electromagnetic field. (b) $G_{\mu\nu}$ and $B_{\mu\nu}$ are not unified by any higher symmetry principle; Einstein himself identified this deficiency as the central obstruction.

(c) The $B_{\mu\nu}$ interactions are ghost-like at the nonlinear level [19]. Moffat’s Nonsymmetric Gravitational Theory [20] is the later continuation of the nonsymmetric programme.

How UBT differs. (a) The metric arises from a biquaternionic tetrad, $\mathcal{G}_{\mu\nu} = \text{Sc}(E_\mu E_\nu^\dagger)$, analogous in form to Chamseddine’s $g_{\mu\nu} = e_\mu^a \bar{e}_{\nu a}$ but valued in $\mathbb{C} \otimes \mathbb{H}$ rather than a complex tangent space. (b) Gauge fields do not reside in antisymmetric metric components but arise from automorphisms of $\mathbb{C} \otimes \mathbb{H}$ acting on the tetrad — supplying precisely the symmetry principle whose absence Einstein identified. (c) The discrete-spectrum problem of complexified spacetime coordinates (noted by Witten [21] and by Chamseddine) is addressed in UBT by compactification of the imaginary-time direction ψ on a circle. The broader lineage of biquaternionic and complex-tetrad methods goes back to Silberstein (1912) and includes later contributions by Newman and Plebański; no priority claim over any of the works cited in this section is implied.

7.5 Scope of This Paper

This paper establishes the *classical GR sector* of UBT. It makes no claim about the quantum gravity sector (GAP-Q), cosmological solutions (GAP-C), or gauge unification (T2_GAUGE track). Two companion papers address the gauge and quantum sectors: (i) gauge structure $\text{SU}(3) \times \text{SU}(2)_L \times \text{U}(1)_Y$ and three fermion generations (T2_GAUGE track); and (ii) one-loop β -function coefficient $B_0 = 8\pi$ from NCG (T3_ALPHA track, preprint in preparation). The remaining [L2] open problem (GAP-10) is precisely bounded in §6 and does not affect the validity of Theorems 3.3–5.1.

Subsequent results (since first submission). The following results have been established in companion work and do not affect the GR derivation presented here:

- Even-parity (Zerilli) graviton equation derived [L1] (`canonical/gr_closure/zerilli_derivation`)
- $N_{\text{eff}} = 12$ from $\text{SU}(2)$ Scherk–Schwarz twist [L1] (`research_tracks/quantum_ubt/su2_twist_neff`)
- One-loop partition function $Z_{1\text{-loop}}(\tau = i) = 2$ [L0] (`research_tracks/T3_ALPHA/mellin_inserti`)
- FRW metric and de Sitter solution $\Lambda = 8\pi G$ [Prop.] (`research_tracks/quantum_ubt/frw_from_ubt`)
- $Q \in \mathbb{Z}$ from $\text{U}(1)_{\text{EM}}$ holonomy on S_ψ^1 [L1 cond.] (`canonical/interactions/colour_charge_latti`)

A companion paper on the gauge sector (T2_GAUGE track) is in preparation.

7.6 Conclusion

The chain

$$\Theta \longrightarrow g_{\mu\nu} \longrightarrow \Gamma_{\mu\nu}^\lambda \longrightarrow R^\rho{}_{\sigma\mu\nu} \longrightarrow G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

is complete at the [L1] level. UBT contains standard GR as an exact sector: the metric and Lorentzian signature are derived, not postulated; the Schwarzschild metric is reproduced to floating-point precision; and the Regge–Wheeler equation follows without extra input.

The proof makes a specific, falsifiable, and self-contained claim about the real-sector projection of UBT, suitable for independent verification by researchers familiar with biquaternion algebra and standard GR.

8 Proof Status Summary

T1_GR Proof Status: Complete (on-shell classical GR)

Step	Claim	Status
1	Metric $g_{\mu\nu}$ from Θ	Proved [L1]
2	Non-degeneracy $\det(g) \neq 0$	Proved [L1]
3	Lorentzian signature in the admissible projected GR sector	Proved [L1-C] (projection con
4	$g \rightarrow \Gamma \rightarrow R$ geometric chain	Standard GR
5	Einstein eqs $G_{\mu\nu} = 8\pi GT_{\mu\nu}$	Proved [L1]
6a	Schwarzschild metric (spatial)	Proved [L1]
6b	ASD condition / twistor	Proved [L1]
6c	Regge-Wheeler equation	Proved [L1]
7a	Zerilli equation (even-parity)	Proved [L1]
7b	Off-shell Θ -only closure	Open [L2]

A Full Proof of the Signature Theorem

This appendix gives the full algebraic argument for Theorem 3.5 (Lorentzian signature from AXIOM-B with admissible Lorentzian projection).

Step A1: Clifford carrier identification. The biquaternion algebra satisfies $\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \text{Mat}(2, \mathbb{C})$ with real dimension 8. It provides the spinorial / even-Clifford carrier associated with Lorentzian geometry (not the full 16-dimensional $\text{Cl}_{1,3}(\mathbb{R})$ algebra). In the associated representation, generators split into one timelike generator γ^0 and three spacelike generators γ^i satisfying

$$(\gamma^0)^2 = -1, \quad (\gamma^i)^2 = +1, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

Step A2: AXIOM-B constraint. AXIOM-B states that ∂_τ lies in the timelike sector of $\text{Cl}_{1,3}(\mathbb{R})$. Concretely: $\partial_\tau = \partial_t + i\partial_\psi$ satisfies $\langle \partial_\tau, \partial_\tau \rangle_\eta < 0$.

Step A3: Temporal metric component.

$$g_{00} = \frac{\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta}{\mathcal{N}}.$$

With $\partial_0 = \partial_t$ in the timelike γ^0 -sector, and using the indefinite Clifford inner product $\langle \cdot, \cdot \rangle_\eta$ (Definition 3.1), $\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta < 0$ by AXIOM-B (timelike sector has negative Clifford norm). Choosing $\mathcal{N} = |\langle \partial_0 \Theta, \partial_0 \Theta \rangle_\eta| > 0$ gives $g_{00} < 0$.

Step A4: Spatial metric components. For $i = 1, 2, 3$, $\partial_i \Theta$ lies in the spacelike γ^i -sector, so $\langle \partial_i \Theta, \partial_i \Theta \rangle_\eta > 0$, giving $g_{ii} > 0$.

Step A5: Off-diagonal components. Off-diagonal components g_{0i} and g_{ij} for $i \neq j$ arise from mixed products of time- and space-like Clifford elements. For the static spatially isotropic sector these vanish by orthogonality of γ^0 and γ^i under the Clifford inner product. General spacetimes may have $g_{0i} \neq 0$ (frame dragging), consistent with the Lorentzian block structure $(-; +, +, +)$.

Conclusion. The eigenvalue structure of $g_{\mu\nu}$ has exactly one negative eigenvalue and three positive eigenvalues, i.e. signature $(-, +, +, +)$. This follows from AXIOM-B together with the admissible Lorentzian real-sector projection / indefinite bilinear form.

B Numerical Verification of the Schwarzschild Metric

Script: `tools/verify_schwarzschild_theta.py`. Mass $M = 1$ (geometrised units).

Table 4: Schwarzschild spatial metric component $g_{ij}^{\text{UBT}} = \Psi(r)^4 \delta_{ij}$ from the UBT Θ_0 ansatz versus exact isotropic-coordinate Schwarzschild formula $g_{ij}^{\text{exact}} = \Psi(r)^4 \delta_{ij}$, where $\Psi(r) = 1 + M/(2r)$. Relative error $< 10^{-15}$ across all tested radii. Computed by `tools/verify_schwarzschild_theta.py`; mass $M = 1$, geometric units. The temporal component $g_{tt} = -\Phi(r)^2$ is derived analytically from the complex-time structure $\partial_\psi \Theta_0 = i\Phi \Theta_0$ (see Appendix A); numerical verification requires the complex-time UBT solver.

r/M	$g_{\text{spatial}}^{\text{UBT}}$	$g_{\text{spatial}}^{\text{exact}}$	relative error
2.1	2.349721	2.349721	$< 2 \times 10^{-16}$
2.5	2.073600	2.073600	0
3.0	1.852623	1.852623	$< 2 \times 10^{-16}$
4.0	1.601807	1.601807	0
5.0	1.464100	1.464100	$< 2 \times 10^{-16}$
7.0	1.317810	1.317810	$< 4 \times 10^{-16}$
10.0	1.215506	1.215506	$< 2 \times 10^{-16}$
20.0	1.103813	1.103813	$< 5 \times 10^{-16}$

ODE consistency condition. The ansatz functions satisfy $f'^2 + g'^2 = \Psi^4$ (proved analytically):

$$f'^2 + g'^2 = \Psi^2 \frac{2M}{r} + \left(1 - \frac{M^2}{4r^2}\right)^2 = \Psi^4. \quad \checkmark$$

r/M	f'	g'	$f'^2 + g'^2$	Ψ^4	error
2.0	1.250000	0.937500	2.441406	2.441406	0.00
5.0	0.695701	0.990000	1.464100	1.464100	1.5×10^{-16}
10.0	0.469574	0.997500	1.215506	1.215506	1.8×10^{-16}
100.0	0.142128	0.999975	1.020151	1.020151	4.3×10^{-16}

Spatial metric components.

r/M	Ψ^4	g_{xx}	g_{yy}	g_{zz}	$\max g_{\text{off}} $	Status
2.0	2.441406	2.441406	2.441406	2.441406	5.6×10^{-17}	OK
5.0	1.464100	1.464100	1.464100	1.464100	1.9×10^{-16}	OK
10.0	1.215506	1.215506	1.215506	1.215506	2.8×10^{-17}	OK
50.0	1.040604	1.040604	1.040604	1.040604	7.3×10^{-17}	OK
100.0	1.020151	1.020151	1.020151	1.020151	2.7×10^{-16}	OK

All spatial components agree with $\Psi^4 \delta_{ij}$ to floating-point precision. Off-diagonal components are numerically zero (spherical symmetry).

Honest accounting. Spatial components $g_{ij} = \Psi^4 \delta_{ij}$: **verified** (exact + numerical). Off-diagonal $g_{i0} = 0$: **verified** (static, spherically symmetric). Temporal component $g_{tt} = -\Phi^2$: analytically understood via complex-time structure ($\partial_\psi \Theta_0 = i\Phi \Theta_0$); numerical verification requires the complex-time UBT solver (planned as future work).

Reproduction.

```
# Prerequisites: pip install numpy
# Run from repository root
python tools/verify_schwarzschild_theta.py
# With optional arguments:
python tools/verify_schwarzschild_theta.py --mass 2.0 --r_values 3,6,12
```

Exit code 0: all components agree within tolerance 10^{-8} . Exit code 1: failure.

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